HOMEWORK 6

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1. Bose-Einstein condensation for $\varepsilon_k = a|\vec{k}|^s$ in d dimensions

Consider an ideal Bose gas with single-particle energy spectrum $\varepsilon_k = a|\vec{k}|^s$ confined to a box in d spatial dimensions (a and s are positive numbers).

a) Under what conditions (a relationship between s and d) does one expect Bose condensation?

b) Show that if Bose condensation does occur the critical temperature is proportional to a power of density, such that $T_c \propto (N/V)^\alpha$, and deduce the exponent α in terms of s and d.

c) Show that if Bose condensation does occur then $C_V \propto (T/T_c)^\gamma$ for $T < T_c$ and deduce the exponent γ in terms of s and d.

2. Isothermal compressibility

Find the Helmholtz free energy and the Gibbs free energy of a gas of free spin-0 bosons in three dimensions. Show that, as the critical temperature is approacha from above, the isothermal compressibility diverges as $\kappa_T \propto (T - T_c)^{-1}$.

3. Internal degree of freedom

An ideal Bose-Einstein gas consists of noninteracting bosons of mass m which have an internal degree of freedom which can be described by assuming, that the bosons are two-level atoms. Bosons in the ground state have energy $E_0 = p^2/2m$, while bosons in the excited state have energy $E_1 = p^2/2m + \Delta$, where p is the momentum and Δ is the excitation energy. Assume that $\Delta \gg k_B T$. Compute the Bose-Einstein condensation temperature, T_c, for this gas of two-level bosons. Does the existence of internal degree of freedom raise or lower the condensation temperature.

4. Entropy of ideal Fermi-Dirac and Bose-Einstein gases

Show that the entropy for an ideal Fermi-Dirac gas (neglecting spin) can be written in the form

$$S = -k_B \sum_l \left[\langle n_l \rangle \ln\langle n_l \rangle + (1 - \langle n_l \rangle) \ln(1 - \langle n_l \rangle)\right].$$

Derive a similar expression for Bose-Einstein statistics.

5. Relativistic Fermi-Dirac gas at zero temperature

Consider a relativistic gas of N particles of spin $1/2$ obeying Fermi statistics, enclosed in volume V, at absolute zero. The energy-momentum relation $\varepsilon_p = \sqrt{(pc)^2 + (mc^2)^2}$, where m is the mass of the particles.

a) Find the Fermi energy at density n.
b) Define the internal energy as the average $\varepsilon_p - mc^2$ and the pressure as the average force per unit area exerted on a perfectly reflecting wall of the container. Set up the expressions for these quantities in the form of integrals, but you need not to evaluate them.
c) Show that $PV = 2U/3$ at low densities and $PV = U/3$ at high densities. State the criteria for low and high densities.

6. Textbook review

Review the relevant sections of the chapters 8, 11 and 12.